ON THE AMPLENESS OF POSITIVE CR LINE BUNDLES OVER LEVI-FLAT MANIFOLDS

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Abstract. We give an example of a compact Levi-flat CR 3-manifold with a positive-along-leaves CR line bundle which is not ample with respect to transversely infinitely differentiable CR sections. This example shows that we cannot improve the regularity of Kodaira type embedding theorem for compact Levi-flat CR manifolds obtained by Ohsawa and Sibony.

1. Introduction

We are going to study function theory on Levi-flat CR 3-manifolds, i.e., 3-manifolds foliated by Riemann surfaces. If we look them as families of Riemann surfaces, we can expect analogy with classical theory on Riemann surfaces, such as the Riemann-Roch theorem. But here is a new ingredient: dynamics of the foliation. We should face subtle interaction between complexity of Levi foliations and existence of CR functions, and especially in the case where the Levi-flat CR 3-manifold is realized as a real hypersurface in a complex surface, it should be reflected on pseudoconvexity of the complement, and complex geometry of the ambient space. There are several attempts toward this direction. We refer the reader to the works of Inaba [15] and Barrett [2].

We investigate such phenomenon in a problem on an analogue of Kodaira's embedding theorem. Ohsawa and Sibony proved the following Kodaira type embedding theorem.

Theorem ([18, Theorem 3], refined in [19])**.** *Let M be a compact $\mathcal{C}^\infty$ Levi-flat CR manifold equipped with a $\mathcal{C}^\infty$ CR line bundle L. Suppose L is positive along leaves, i.e., there exists a $\mathcal{C}^\infty$ hermitian metric on L such that the restriction of the curvature form to each leaf is everywhere positive definite. Then, for any $\kappa \in \mathbb{N}$, L is $\mathcal{C}^\kappa$-ample, i.e., there exists $n_0 \in \mathbb{N}$ such that one can find CR sections $s_0, \cdots, s_N$ of $L^{\otimes n}$, of class $\mathcal{C}^\kappa$, for any $n \geq n_0$, such that the ratio $(s_0 : \cdots : s_N)$ embeds M into $\mathbb{CP}^N$.*

We can make the regularity $\kappa \in \mathbb{N}$ arbitrarily large although we need to take n_0 sufficiently large. A natural question is whether or not we can improve the regularity to $\kappa = \infty$. The answer is no, in general, as the following case-study tells us.

Main Theorem. *Let Σ be a compact Riemann surface, and $\mathcal{D}$ a holomorphic disc bundle over Σ. Denote its associated compact $\mathcal{C}^\infty$ Levi-flat CR manifold by $M = \partial\mathcal{D}$*

Date: January 28, 2013.

2010 *Mathematics Subject Classification.* Primary 32V30, Secondary 32E10, 32V25, 53D35.

Key words and phrases. projective embedding, Levi-flat CR manifold, holomorphic disc bundle, pseudoconvexity, confoliation.

in its associated flat ruled surface $\pi\colon X \to \Sigma$. Take a positive line bundle L over Σ. Suppose $\mathcal{D}$ has a unique non $\pm$holomorphic harmonic section, then $\pi^*L|M$ is positive along leaves, but never $\mathcal{C}^\infty$ ample.

It is easily checked the point that the pull-back bundle $\pi^*L|M$ is positive along leaves. Thus, this theorem states non $\mathcal{C}^\infty$-ampleness of such CR line bundles.

The assumption is fulfilled for the following explicit example (Example 3.2): Let Σ be a compact Riemann surface of genus ≥ 2. Fix an identification of a universal covering $\widetilde{\Sigma} \simeq \mathbb{D}$ and regard $\pi_1(\Sigma) \simeq \Gamma < \mathrm{Aut}(\mathbb{D})$ as the Fuchsian representation of Σ. Take a non-trivial quasiconformal deformation of Γ, say $\rho\colon \Gamma \to \mathrm{Aut}(\mathbb{D})$. Set $\mathcal{D} := \widetilde{\Sigma} \times \mathbb{D}/(z,\zeta) \sim (\gamma z, \rho(\gamma)\zeta)$ for $\gamma \in \Gamma$.

Another research direction of the analogue of Kodaira's embedding theorem is the problem concerning on projective embedding of compact laminations. We can find similar phenomenon in the work of Fornæss and Wold [11, Theorem 5.1] where they study compact $\mathcal{C}^1$-smooth hyperbolic laminations. We also refer the reader to the works of Gromov [13, pp.401–402], Ghys [12, §7] and Deroin [7].

The organization of the paper is as follows. In §2, we introduce basic notions on Levi-flat CR manifolds. In §3, we recall and refine a classification result of holomorphic disc bundles with an emphasis on *Takeuchi 1-completeness* of certain holomorphic disc bundles. This notion is also known as $\log\delta$-pseudoconvexity [3] or strong Oka property [14], and would be of interest from the viewpoint of con-foliation. In §4, we state a variant of Bochner-Hartogs type extension theorem for CR sections. We give a self-contained proof for the reader's convenience. In §5, we prove Main Theorem and pose some further questions.

2. Preliminaries

We explain the notion of Levi-flat CR manifolds. For simplicity, we discuss under the assumption that manifolds and bundles have at least $\mathcal{C}^\infty$-smoothness.

2.1. **Almost complex structure.**

We briefly recall almost complex structure and its relation with complex structure as a preparation for the subsequent subsections.

Definition 2.1 (almost complex structure)**.** Let X be a real $2n$-dimensional $\mathcal{C}^\infty$ manifold. An *almost complex structure* on X is $J \in \mathrm{End}(TX)$ satisfying $J^2 = -\mathrm{Id}$.

By using the J, we can define an action $\mathbb{C} \times TX \ni (a+bi, v) \mapsto (a+bJ)v \in TX$, which enables us to regard TX as a complex vector bundle of $\mathrm{rank}_{\mathbb{C}} = n$.

We can easily see that J is diagonalizable on the complexified tangent bundle $\mathbb{C} \otimes TX$ with eigenvalues $\pm i$. Denote the eigenvalue decomposition by $\mathbb{C} \otimes TX = T^{1,0}X \oplus T^{0,1}X$ where $T^{1,0}X := \mathrm{Ker}(J - i\mathrm{Id})$ (the *holomorphic tangent bundle* of X) and $T^{0,1}X := \overline{T^{1,0}X} = \mathrm{Ker}(J + i\mathrm{Id})$ (the *anti-holomorphic tangent bundle* of X). We identify TX with $T^{1,0}X$ as complex vector bundle by $TX \ni v \mapsto (v - iJv)/2 \in T^{1,0}X$.

Any complex manifold is equipped with its canonical almost complex structure given by
$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$
where $(z_j = x_j + iy_j)_{j=1,\cdots,n}$ is any holomorphic local coordinate. On the other hand, not every almost complex structure comes from a complex structure. The following Newlander-Nirenberg theorem gives a criterion:

Theorem 2.2 (Newlander-Nirenberg)**.** *An almost complex structure J comes from a complex structure if and only if smooth sections of the holomorphic tangent bundle with respect to J are closed under the Lie bracket.*

Here, the Lie bracket means the complex linear extension of the usual Lie bracket of vector fields.

2.2. **Levi-flatness in terms of foliation.** We give a precise definition of Levi-flat CR manifold. First recall the definition of foliation:

Definition 2.3 (foliation)**.** Let M be a real m-dimensional $\mathcal{C}^\infty$ manifold. A $\mathcal{C}^\infty$ *foliation* on M of real codimension d is a decomposition of M into arcwise-connected injectively immersed submanifolds $\mathcal{F} = \{N_\alpha\}_{\alpha\in\Lambda}$ of real codimension d in such a way that for any $p \in M$ we can find a chart $\varphi\colon U \to \mathbb{R}^{m-d} \times \mathbb{R}^d$ around p so that for any arcwise-connected component P of $N_\alpha \cap U$ $(\alpha \in \Lambda)$ has a unique $t \in \mathbb{R}^d$ such that $P = \varphi^{-1}(\mathbb{R}^{m-d} \times \{t\})$.

We call N_α a *leaf*, φ a *foliated chart*, and P a *plaque*. By abuse of notation, we do not distinguish the immersion of a leaf $\iota_\alpha\colon N_\alpha \hookrightarrow M$ with its image. However, if we say *leafwise*, we consider something not only to be restricted on each leaf but also to be discussed in the *leaf topology*, i.e., in the topology of N_α, the domain of ι_α, not in the induced topology of $\iota_\alpha(N_\alpha) \subset M$.

We collect vectors tangent to leaves, and form them into a subbundle $T\mathcal{F}$ of TM, whose local triviality is assured by the requirement in the definition of foliation. We call $T\mathcal{F}$ the *tangent bundle* of $\mathcal{F}$. The following Frobenius theorem tells us when we can recover a foliation by a given subbundle:

Theorem 2.4 (Frobenius)**.** *Suppose a subbundle D of TM is given. There is a foliation $\mathcal{F}$ whose tangent bundle $T\mathcal{F}$ equals to the given subbundle D. if and only if smooth sections of D are closed under the Lie bracket,*

Using these terminologies and the view of the Newlander-Nirenberg theorem, we give the following definition.

Definition 2.5 (Levi-flat CR manifold)**.** A $\mathcal{C}^\infty$ *Levi-flat CR manifold* is a triple $(M, \mathcal{F}, J)$ where M is a $\mathcal{C}^\infty$ manifold, $\mathcal{F}$ is a $\mathcal{C}^\infty$ foliation on M of real codimension one (the *Levi foliation* of M), and J is a $\mathcal{C}^\infty$ section of $\mathrm{End}(T\mathcal{F})$ that induces a complex structure on each leaf, i.e., $J^2 = -\mathrm{Id}$ and smooth sections of $T^{1,0} := \mathrm{Ker}(J - i\mathrm{Id}) \subset \mathbb{C} \otimes T\mathcal{F} \subset \mathbb{C} \otimes TM$ are closed under the Lie bracket.

The simplest example is $M = \mathbb{C}^{n-1} \times \mathbb{R}$ where the foliation is given by its leaves $\{\mathbb{C}^{n-1} \times \{t\}\}_{t\in\mathbb{R}}$ and J is induced from the standard complex structure of $\mathbb{C}^{n-1}$. This provides the local structure of Levi-flat CR manifolds under the requirement for foliated charts to be leafwise holomorphic with respect to J. In other words, any $\mathcal{C}^\infty$ Levi-flat CR manifold can be constructed by gluing some open sets of $\mathbb{C}^{n-1} \times \mathbb{R}$ together using leafwise holomorphic $\mathcal{C}^\infty$ maps.

Our object of study can be defined as follows:

Definition 2.6 (CR function)**.** We say that a function $f\colon M \to \mathbb{C}$ is a *CR function* if it is leafwise holomorphic.

2.3. **Levi-flatness in terms of CR geometry.** We will investigate Levi-flat CR manifolds embedded in complex manifolds, especially Levi-flat real hypersurfaces. The following reformulation in terms of CR geometry is suitable for this purpose.

Let us start with the definition of general CR manifold.

Definition 2.7 (CR manifold)**.** A *CR manifold* (*of hypersurface type*) is a pair $(M, T^{1,0})$ where M is a $\mathcal{C}^\infty$ manifold of dimension $2n-1$, and $T^{1,0}$ is a subbundle of $\mathbb{C} \otimes TM$ of $\mathrm{rank}_{\mathbb{C}}\ n-1$. Denote $T^{0,1} := \overline{T^{1,0}}$. We require that $T^{1,0} \cap T^{0,1} = 0$ and smooth sections of $T^{1,0}$ are closed under the Lie bracket.

It models a real hypersurface M of an n-dimensional complex manifold (X, J_X), for such M we can put $T^{1,0} := T^{1,0}X \cap \mathbb{C}TM \simeq$ (the maximal J_X-invariant subspace of TM). Moreover, if the real hypersurface M is given by a $\mathcal{C}^\infty$ defining function r, namely, $r\colon M \subset U \to \mathbb{R}$ with $M = \{z \in U \mid r(z) = 0\}$ and $dr \neq 0$ on M, we have $T^{1,0} = \mathrm{Ker}\partial r \subset T^{1,0}X$.

Now we can redefine

Definition 2.8 (Levi-flat CR manifold)**.** A $\mathcal{C}^\infty$ *Levi-flat CR manifold* is a $\mathcal{C}^\infty$ CR manifold $(M, T^{1,0})$ such that smooth sections of $T^{1,0} + \overline{T^{1,0}}$ are closed under the Lie bracket.

Its foliation $\mathcal{F}$ is recovered by integrating the distribution $(T^{1,0} + \overline{T^{1,0}}) \cap TM$ thanks to the Frobenius theorem, and its leafwise complex structure J is recovered by the CR structure $T^{1,0}$ thanks to the requirement for CR structure and the Newlander-Nirenberg theorem. In the case that M is located in a complex manifold X with defining function r, M is Levi-flat if and only if its Levi form $i\partial\overline{\partial}r|T^{1,0} = 0$ as a quadratic form. This is the classical definition of *Levi-flat real hypersurface.*

We can also redefine our functions:

Definition 2.9 (CR function)**.** We say that a function $f\colon M \to \mathbb{C}$ is a *CR function* if it is annihilated by vectors of $T^{0,1}$.

If $M = \{r = 0\} \subset X$ and f is of $\mathcal{C}^1$, it is equivalent to say that $\overline{\partial}\tilde{f}$ is proportional to $\overline{\partial}r$ on M where $\tilde{f}$ is any $\mathcal{C}^1$ extension of f on a neighborhood of M. In particular, the restriction of any holomorphic function defined near M is CR.

2.4. **Line bundles.** We clarify the definition of curvature for CR line bundles over Levi-flat CR manifolds, and remark an important example of line bundle.

Definition 2.10 (CR line bundle)**.** A $\mathcal{C}^\infty$ *CR line bundle* over a $\mathcal{C}^\infty$ Levi-flat CR manifold M is a $\mathcal{C}^\infty$ complex vector bundle of $\mathrm{rank}_{\mathbb{C}}$ 1 that possesses a trivialization cover whose transition functions are CR.

A straightforward example of CR line bundle is the restriction of a holomorphic line bundle on a Levi-flat real hypersurface.

Now let h be a $\mathcal{C}^\infty$ hermitian metric on L. In the case of holomorphic line bundles over complex manifolds, we can induce the Chern connection and its curvature from the metric h . But in the case of CR line bundles over Levi-flat CR manifolds, since our complex structure is defined only for the leaf direction, we cannot define such canonical connection and curvature. But still, we can define the curvature along leaves canonically.

Proposition 2.11 (curvature along leaves)**.** *Let D be any connection on L that agrees with the Chern connection on $(L|N, h)$ along any leaf N. Then, Θ_h the curvature 2-form of the connection D restricted along $T\mathcal{F}$ is independent of the choice of D.*

The reason is that we have the local expression $\Theta_h = -\partial_z\overline{\partial}_z \log h(z,t)$ where $(z,t)\colon M \supset U \to \mathbb{C}^{n-1} \times \mathbb{R}$ is any foliated chart. This notion of curvature justifies the following terminology.

Definition 2.12 (positive along leaves)**.** We say a CR line bundle L to be *positive along leaves* if there exists a hermitian metric h on L whose curvature along leaves determines a positive definite quadratic form everywhere.

For compact Levi-flat CR 3-manifolds, the existence of a positive-along-leaves bundle imposes the following restriction on the topology of its Levi foliation.

Proposition 2.13. *A compact $\mathcal{C}^\infty$ Levi-flat CR 3-manifold $(M, \mathcal{F}, J)$ possesses a $\mathcal{C}^\infty$ CR line bundle which is positive along leaves if and only if the Levi foliation $\mathcal{F}$ is taut.*

Definition 2.14. A $\mathcal{C}^\infty$ foliation $\mathcal{F}$ of real codimension one on a $\mathcal{C}^\infty$ manifold M is *taut* if there exists a $\mathcal{C}^1$ closed transversal, i.e., a $\mathcal{C}^1$ embedded circle in M which transversely intersects every leaf of $\mathcal{F}$.

We will use the following geometric characterization of tautness to prove Proposition 2.13.

Theorem 2.15 (Rummler [21], Sullivan [23])**.** *A $\mathcal{C}^\infty$ foliation $\mathcal{F}$ of real codimension one on a closed $\mathcal{C}^\infty$ manifold M is taut if and only if there exists a $\mathcal{C}^2$ Riemannian metric on M with respect to which every leaf of $\mathcal{F}$ is minimal.*

Proof of Proposition 2.13. Suppose that M possesses a positive-along-leaves bundle. Ohsawa-Sibony's embedding theorem implies that M can be $\mathcal{C}^2$ CR embedded in a complex projective space. We put a Riemannian metric on M by restricting the Fubini-Study metric. Then, any leaf of $\mathcal{F}$ is minimal since any complex submanifold in a Kähler manifold is minimal with respect to its Kähler metric.

Conversely, suppose that M is taut. By smoothing a closed transversal, we have a $\mathcal{C}^\infty$ one, say T. Regarding the intersection of T with the leaves of $\mathcal{F}$ as a divisor, we can construct a positive-along-leaves $\mathcal{C}^\infty$ CR line bundle. □

We close this section with an important example of line bundle.

Example 2.16. Let M be a $\mathcal{C}^\infty$ Levi-flat CR manifold. We can define the *normal bundle* $N_\mathcal{F}$ of the Levi foliation $\mathcal{F}$ by $N_\mathcal{F} := \mathbb{C} \otimes TM/T\mathcal{F}$. It is easy to check that $N_\mathcal{F}$ is a CR line bundle as follows:

Proof. We can trivialize it by $\partial/\partial t$ on a foliated chart $(z,t)\colon U \to \mathbb{C}^{n-1} \times \mathbb{R}$. If we have two intersecting foliated charts, say $(z,t), (z',t')$, we know that t' depends only on t not on z. Thus, the transition function of the normal bundle $\partial t'/\partial t$ is constant function in z, especially CR. □

When the Levi-flat CR manifold is realized as a Levi-flat real hypersurface M in a complex manifold X, we have another definition of the normal bundle: define the *complex normal bundle* of M by a quotient of CR vector bundles $N^{1,0}_M :=$

$T^{1,0}X/T^{1,0}$. It follows easily that the two normal bundles $N_{\mathcal{F}}$ and $N_M^{1,0}$ are isomorphic as CR line bundle.

An important feature of this normal bundle is that it simultaneously approximates both transverse structure of the Levi foliation $\mathcal{F}$ and neighborhood of M in X, which feature permits us to take the viewpoint: dynamical property of the Levi foliation $\mathcal{F}$ is reflected on pseudoconvexity of $X \setminus M$. For this direction, we refer the reader to the work of Brunella [4].

3. Holomorphic disc bundles in flat ruled surfaces

We recall a classification result on holomorphic disc bundles, with which a standard example of Levi-flat CR 3-manifolds associate, and supplement preceding results about pseudoconvexity of these spaces.

3.1. **Holomorphic disc bundles.** We begin by recalling a construction of holomorphic disc bundles. Let Σ be a compact Riemann surface. A holomorphic fiber bundle over Σ with fiber $\mathbb{D} := \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ is called a *holomorphic disc bundle* over Σ. It can be easily seen that holomorphic trivializations form a flat trivializing cover, i.e., all of the transition functions are locally constant.

Hence, any holomorphic disc bundle $\mathcal{D}$ can be obtained by the *suspension* construction: we can find a group homomorphism $\rho\colon \pi_1(\Sigma) \to \mathrm{Aut}(\mathbb{D})$, called a *holonomy homomorphism*, giving a bundle isomorphism

$$\begin{aligned} \mathcal{D} &\simeq \Sigma \times_\rho \mathbb{D} \\ &:= \widetilde{\Sigma} \times \mathbb{D}/(z,\zeta) \sim (\gamma z, \rho(\gamma)\zeta) \text{ for } \gamma \in \pi_1(\Sigma) \end{aligned}$$

where $\widetilde{\Sigma}$ is a universal covering of Σ. We denote this disc bundle by $\mathcal{D}_\rho$.

The group $\mathrm{Aut}(\mathbb{D})$ of biholomorphisms of $\mathbb{D}$ consists of Möbius transformations preserving $\mathbb{D}$, acting on the Riemann sphere $\mathbb{CP}^1$ and fixing the unit circle $\partial\mathbb{D}$. Thus, it follows that a holomorphic disc bundle is canonically embedded in its associated flat ruled surface, say $\pi\colon X_\rho := \Sigma \times_\rho \mathbb{CP}^1 \to \Sigma$, and the boundary of $\mathcal{D}_\rho$ in X_ρ, a flat circle bundle, becomes a compact $\mathcal{C}^\omega$ Levi-flat CR 3-manifold, say $M_\rho := \Sigma \times_\rho \partial\mathbb{D}$. Note that $\mathcal{D}_\rho \to X \setminus \overline{\mathcal{D}_\rho}$, $(z,\zeta) \mapsto (z, 1/\overline{\zeta})$ is an anti-biholomorphism, which we call the *conjugation*.

3.2. **Classification.** Now we state a classification result of holomorphic disc bundles by means of harmonic sections:

Theorem 3.1 ([8], [9, Proposition 1.1])**.** *Let $\mathcal{D}$ be a holomorphic disc bundle over a compact Riemann surface and M its associated Levi-flat CR 3-manifold. Then, one of the following cases occurs:*

(i) *$\mathcal{D}$ admits a unique non-holomorphic harmonic section.*
(ii) *$\mathcal{D}$ admits a unique locally non-constant holomorphic section.*
(iii) *M admits one or two locally constant section(s).*
(iv) *$\mathcal{D}$ admits a locally constant section.*

Here a section is said to be *harmonic* if it is lifted to a ρ-equivariant harmonic map $\tilde{h}\colon \widetilde{\Sigma} \to \mathbb{D}$ where $\mathbb{D}$ is equipped with the Poincaré metric, and a section is said to be *locally constant* if it is locally constant in the (flat) trivializing coordinates.

Example 3.2. We describe examples of each case in terms of holonomy homomorphism.

(i) Let Σ be of genus ≥ 2. Fix an identification $\widetilde{\Sigma} \simeq \mathbb{D}$ and regard $\pi_1(\Sigma) \simeq \Gamma < \mathrm{Aut}(\mathbb{D})$ as the Fuchsian representation of Σ. Take a non-trivial quasiconformal deformation of Γ, say $\rho\colon \Gamma \to \mathrm{Aut}(\mathbb{D})$. Then $\mathcal{D}_\rho$ is of the case (i). The unique harmonic section corresponds to the graph of the unique harmonic diffeomorphism $\Sigma = \mathbb{D}/\Gamma \to \mathbb{D}/\rho(\Gamma)$.

(ii) Let Σ and Γ as above, and $\rho = \mathrm{Id}\colon \Gamma \to \Gamma \subset \mathrm{Aut}(\mathbb{D})$. Then, $\mathcal{D}_\rho$ is of the case (ii). Its associated holomorphic section is obtained by the quotient of the diagonal set $\Delta \subset \widetilde{\Sigma} \times \mathbb{D} = \mathbb{D} \times \mathbb{D}$.

(iii) Let ρ be a homomorphism from $\pi_1(\Sigma)$ to an abelian subgroup of $\mathrm{Aut}(\mathbb{D})$ that consists of parabolic (resp. hyperbolic) elements with common fixed point(s) on $\partial\mathbb{D}$. Then, $\mathcal{D}_\rho$ is of the case (iii). The locally constant section(s) correspond(s) to the suspension of the fixed point(s).

(iv) Let ρ be a homomorphism from $\pi_1(\Sigma)$ to an abelian subgroup of $\mathrm{Aut}(\mathbb{D})$ that consists of elliptic elements with common fixed point in $\mathbb{D}$, which is just isomorphic to the group of rotations $\mathrm{U}(1)$. Then, $\mathcal{D}_\rho$ is of the case (iv). The suspension of the fixed point gives a locally constant section.

For the cases (i) and (ii), we described only the cases where associated flat circle bundles M_ρ attain the maximal Euler number in the Milnor-Wood inequality. The Euler number of a flat circle bundle M_ρ over Σ, say $\chi(M_\rho)$, satisfies the Milnor-Wood inequality: $|\chi(M_\rho)| \leq \max\{0, 2\mathrm{genus}(\Sigma) - 2\}$. The component where $\chi(M_\rho) = 2\mathrm{genus}(\Sigma) - 2$ is naturally identified with the Teichmüller space of Σ via quasiconformal deformation of Fuchsian representation.

For the cases (iii) and (iv), the examples above exhaust the cases, respectively. They all belong to the component where $\chi(M_\rho) = 0$.

3.3. **Pseudoconvexity.** We prepare several definitions in order to express pseudoconvexity of the domain bounded by a Levi-flat CR manifold, on which dynamics of the Levi foliation is reflected.

First recall some terminologies on potential function.

Definition 3.3. Let X be a complex manifold of dimension n and $\psi\colon X \to [-\infty, \infty)$. We say that ψ is

- *plurisubharmonic* if it is upper semicontinuous and its restriction $\psi|C$ to any holomorphic curve $C \hookrightarrow X$ is subharmonic.
- an *exhaustion* function (resp. a *bounded exhaustion* function) if $\sup_X \psi = \infty$ (resp. $\sup_X \psi < \infty$) and for any $c \in (-\infty, \sup_X \psi)$ the sublevel set $\{z \in X \mid \psi(z) < c\}$ is relatively compact in X.

Suppose that $\psi\colon X \to (-\infty, \infty)$ and ψ is of $\mathcal{C}^2$. Define its *Levi form* as the quadratic form determined by $i\partial\overline{\partial}\psi$. We say that ψ is

- *q-convex* if its Levi from has at least $(n - q + 1)$ positive eigenvalues everywhere.
- *weakly q-convex* if its Levi from has at least $(n - q + 1)$ nonnegative eigenvalues everywhere.
- *strictly plurisubharmonic* if its Levi form is positive definite everywhere, that is, 1-convex.

Let us define several notions of pseudoconvexity, which express what kind of potential functions a domain carries.

Definition 3.4. Let X be a complex manifold of dimension n. We say that X is

- *pseudoconvex* if X possesses a continuous plurisubharmonic exhaustion function.
- *q-convex* (resp. *weakly q-convex*) if X possesses a $\mathcal{C}^\infty$ exhaustion function which is q-convex (resp. weakly q-convex) outside a compact set of X.
- *q-complete* (resp. *weakly q-complete*) if X possesses a $\mathcal{C}^\infty$ q-convex (resp. weakly q-convex) exhaustion function.
- *hyperconvex* if X possesses a $\mathcal{C}^\infty$ strictly plurisubharmonic bounded exhaustion function.

Note that 1-completeness is equivalent to being Stein.

The classical pseudoconvexities above do not ask growth order of the potential function along the boundary, or boundary behavior of eigenvalues of its Levi form. We follow the following definition in [9].

Definition 3.5 (Takeuchi q-convex space)**.** Let X be a complex manifold of dimension n and D a relatively compact domain in X with $\mathcal{C}^2$ boundary. D is said to be *Takeuchi q-convex* if there exists a $\mathcal{C}^2$ defining function r of ∂D defined on a neighborhood of D with $D = \{z \mid r(z) < 0\}$ such that, with respect to a hermitian metric on X, at least $n-q+1$ eigenvalues of the Levi form of $-\log(-r)$ are greater than 1 outside a compact set of D.

Potential functions having the particular form $-\log(-r)$ have their origin in Oka [20], where the Levi problem was solved for domains in $\mathbb{C}^2$ by using the potential function $-\log$ (the Euclidean distance to its boundary).

To return to our case, holomorphic disc bundles, known facts on pseudoconvexity of them are summarized as follows:

- In all the cases, $\mathcal{D}$ is weakly 1-complete ([8, Theorem 1]).
- In the cases (i)–(iii), $\mathcal{D}$ is 1-convex. It is particularly 1-complete, i.e., Stein in the cases (i) and (iii) ([2, Theorem 2]).
- In the cases (i) and (ii), $\mathcal{D}$ is Takeuchi 1-convex ([9, Proposition 1.6][1]).

We will give a supplemental result for the case (i) using the following notion.

Definition 3.6 (Takeuchi q-complete space)**.** Let X be a complex manifold of dimension n and D a relatively compact domain in X with $\mathcal{C}^2$ boundary. D is said to be *Takeuchi q-complete* if there exists a $\mathcal{C}^2$ defining function r of ∂D defined on a neighborhood of D with $D = \{z \mid r(z) < 0\}$ such that, with respect to a hermitian metric on X, at least $n-q+1$ eigenvalues of the Levi form of $-\log(-r)$ are greater than 1 entire on D.

This notion originates in the work of Takeuchi [24] where he showed any proper locally pseudoconvex domain in $\mathbb{CP}^n$ acquires this property for $q = 1$. Although it has already had other names, $\log\delta$-pseudoconvexity in [3], and the strong Oka condition in [14], we name it again in consideration of consistency with the terms in Definition 3.4 and 3.5.

Takeuchi 1-completeness not only implies that the domain is Stein, but also implies that it behaves as if it is in complex Euclidean space:

Theorem 3.7 ([16, Theorem 1.1])**.** *Let D be a Takeuchi 1-complete domain with defining function r. Then, $-\partial\overline{\partial}\log(-r)$ gives a complete Kähler metric on D, and*

[1] Its proof seems to contain some errors.

it follows that $-(-r)^{t_0}$ with sufficiently small $t_0 > 0$ becomes a strictly plurisubharmonic bounded exhaustion function on D, i.e., D is hyperconvex.

Remark 3.8. From the viewpoint of confoliation [10, Corollary 1.1.10], we can translate a question on various strong pseudoconvexity of the complement of a Levi-flat real hypersurface into one on approximation of a foliation by contact structures. For example, suppose a compact Levi-flat real hypersurface M has a Takeuchi 1-convex complement with defining function r. For small positive ε, the level sets $\{r = -\varepsilon\}$ are diffeomorphic to M and possess contact structures induced from the strictly pseudoconvex CR structures. Thus, the family of the level sets defines a "uniform" contact deformation of the Levi foliation. Here "uniform" means that convergence to the foliation is *exactly* the same order entire on M.

3.4. **Takeuchi 1-complete case.** We give the following supplemental result, which is the main technical point of this paper, on pseudoconvexity of holomorphic disc bundles for the case (i) in Theorem 3.1.

Proposition 3.9. *Let $\mathcal{D}$ be a holomorphic disc bundle over a compact Riemann surface Σ with a uniquely determined non-holomorphic harmonic section h. Then, $\mathcal{D}$ is Takeuchi 1-complete in its associated ruled surface X.*

Proof. Fix a finite open covering $\{U_\nu\}$ of Σ giving trivializations of $\mathcal{D}$. Set $\delta = \max_\nu \sup_{U_\nu} |h| < 1$ where the value of h is taken with respect to the trivializing coordinate over each U_ν. It suffices to find a defining function r of $\partial\mathcal{D}$ so that the eigenvalues of the complex Hessian of $-\log(-r)$ in each trivializing coordinate $(z, \zeta)\colon \pi^{-1}(U_\nu) \to \mathbb{C}^2$ are bounded from below by a positive constant, since we can easily find a hermitian metric on X that is comparable to $i(dz d\overline{z} + d\zeta d\overline{\zeta})$ by usual "partition of unity" argument.

We will find the desired r in the form $r = r_0 e^{-\psi}$ where r_0 is the defining function of $\partial\mathcal{D}$ used in [8], and $\psi\colon \Sigma \to \mathbb{R}$ will be determined later. Recall the original defining function

$$r_0(z, \zeta) := \left| \frac{\zeta - h(z)}{1 - \overline{h(z)}\zeta} \right|^2 - 1$$

where (z, ζ) is any trivializing coordinate. It is clearly well-defined since the term inside the modulus is just a Möbius transformation that maps $h(z)$ to 0 and remaining choices of the fiber coordinate are only up to rotations.

Take one of the trivializations, say $(z, \zeta)\colon \pi^{-1}(U_\nu) \to \mathbb{C}^2$. The Levi form is

$$\begin{aligned}
&i\partial\overline{\partial}(-\log(-r)) \\
&= i\partial\overline{\partial}\left(\psi - \log(1 - |\zeta|^2) - \log(1 - |h|^2) + 2\mathrm{Re}\log(1 - \overline{h}\zeta)\right) \\
&= \left(\psi_{z\overline{z}} + (1 - |\zeta|^2)(|h_z|^2 + |h_{\overline{z}}|^2) + |\zeta - h|^2 |h_z - e^{2i\theta(z,\zeta)}\overline{h_{\overline{z}}}|^2\right) \frac{idz \wedge d\overline{z}}{|1 - \overline{h}\zeta|^2 (1 - |h|^2)} \\
&\qquad - h_z \frac{idz \wedge d\overline{\zeta}}{(1 - h\overline{\zeta})^2} - \overline{h_z} \frac{id\zeta \wedge d\overline{z}}{(1 - \overline{h}\zeta)^2} + \frac{id\zeta \wedge d\overline{\zeta}}{(1 - |\zeta|^2)^2}
\end{aligned}$$

where $\theta(z, \zeta) := \arg(\zeta - h)/(1 - \overline{h}\zeta)$ and all the values on h and ψ are taken at z. We can check it by direct computation in three steps:

(i) Fix $z_0 \in U$ in the trivialization. Choose a temporal trivializing coordinate $(z, \zeta^\natural)$ with $h^\natural(z_0) = 0$.

(ii) Compute the Levi form on the fiber $\mathcal{D}_{z_0}$ in $(z, \zeta^\natural)$ coordinate. Note that the harmonicity of h yields $h_{z\overline{z}}(z_0) = 0$.
(iii) Pull back the form to (z, ζ) coordinate.

Now we are going to estimate the eigenvalues of the complex Hessian. The trace and determinant of the complex Hessian of $-\log(-r)$ are estimated as

$$\begin{aligned}
\text{trace} &= \frac{1}{(1-|\zeta|^2)^2} + \frac{\psi_{z\overline{z}} + (1-|\zeta|^2)(|h_z|^2 + |h_{\overline{z}}|^2) + |\zeta - h|^2 |h_z - e^{2i\theta(z,\zeta)}\overline{h_{\overline{z}}}|^2}{|1-\overline{h}\zeta|^2(1-|h|^2)} \\
&\le \frac{1}{(1-|\zeta|^2)^2} + \frac{\psi_{z\overline{z}} + (1-|\zeta|^2)(|h_z|^2 + |h_{\overline{z}}|^2) + |\zeta - h|^2 |h_z - e^{2i\theta(z,\zeta)}\overline{h_{\overline{z}}}|^2}{(1-\delta)^3} \\
&\le \frac{1}{(1-|\zeta|^2)^2} + \frac{\psi_{z\overline{z}} + (1-|\zeta|^2 + 2|\zeta-h|^2)(|h_z|^2 + |h_{\overline{z}}|^2)}{(1-\delta)^3} \\
&\le \frac{1}{(1-|\zeta|^2)^2} + \frac{\psi_{z\overline{z}} + 8(|h_z|^2 + |h_{\overline{z}}|^2)}{(1-\delta)^3} \\
&\le \frac{1}{(1-|\zeta|^2)^2} + \sup_U \frac{\psi_{z\overline{z}} + 8(|h_z|^2 + |h_{\overline{z}}|^2)}{(1-\delta)^3} =: \frac{1}{(1-|\zeta|^2)^2} + C.
\end{aligned}$$

$$\begin{aligned}
\det &= \frac{\psi_{z\overline{z}}}{(1-|\zeta|^2)^2} + \frac{1}{(1-|\zeta|^2)^2}\left(\frac{|\zeta-h|^2|h_z - e^{2i\theta(z,\zeta)}\overline{h_{\overline{z}}}|^2}{|1-\overline{h}\zeta|^2(1-|h|^2)^2}\right) \\
&\quad + \frac{1}{1-|\zeta|^2}\left(\frac{|h_{\overline{z}}|^2}{|1-\overline{h}\zeta|^2(1-|h|^2)} + \frac{|\zeta-h|^2|h_z|^2}{|1-\overline{h}\zeta|^4(1-|h|^2)}\right) \\
&\ge \frac{\psi_{z\overline{z}}}{(1-|\zeta|^2)^2} + \frac{|\zeta-h|^2|h_z - e^{2i\theta(z,\zeta)}\overline{h_{\overline{z}}}|^2}{4(1-|\zeta|^2)^2} + \frac{|h_{\overline{z}}|^2}{4(1-|\zeta|^2)} + \frac{|\zeta-h|^2|h_z|^2}{16(1-|\zeta|^2)} \\
&\ge \frac{\psi_{z\overline{z}}}{(1-|\zeta|^2)^2} + \frac{|\zeta-h|^2(|h_z| - |h_{\overline{z}}|)^2 + (1-|\zeta|^2)|h_{\overline{z}}|^2}{4(1-|\zeta|^2)^2} \\
&\ge \frac{\psi_{z\overline{z}}}{(1-|\zeta|^2)^2} + \frac{(|\zeta-h|^2 + 1 - |\zeta|^2)\min\{(|h_z| - |h_{\overline{z}}|)^2, |h_{\overline{z}}|^2\}}{4(1-|\zeta|^2)^2} \\
&\ge \frac{\psi_{z\overline{z}}}{(1-|\zeta|^2)^2} + \frac{(1-\delta)^2\min\{(|h_z| - |h_{\overline{z}}|)^2, |h_{\overline{z}}|^2\}}{4(1-|\zeta|^2)^2}.
\end{aligned}$$

We will set ψ to have sufficiently small range so that the trace is positive, in which situation the smaller eigenvalue λ of the complex Hessian of $-\log(-r)$ is estimated as

$$\begin{aligned}
\lambda &= \frac{\text{trace}}{2} - \sqrt{\frac{\text{trace}}{2} - \det} \ge \frac{\det}{\text{trace}} \\
&\ge \frac{1}{1+C}\left(\psi_{z\overline{z}} + \frac{(1-\delta)^2}{4}\min\{|h_{\overline{z}}|^2, (|h_z| - |h_{\overline{z}}|)^2\}\right).
\end{aligned}$$

Note that this estimate does not depend on ζ, and is sharp in the sense that the smaller eigenvalue of the complex Hessian of $-\log(-r_0)$, which corresponds to the second term in the estimate, actually equals to 0 at $(z, 0)$ if $h_{\overline{z}}(z) = 0$ and tends to 0 near some points of $\partial\mathcal{D}_z$ if $|h_z(z)| = |h_{\overline{z}}(z)|$, which facts can be deduced from the explicit formula of the Levi form. This situation leads us to modify r_0 with ψ strictly subharmonic on such locus in Σ.

From Lemma 3.10 below, we can find a non-empty relatively compact set $V \subset \Sigma$ on which both $|h_{\overline{z}}|$ and $|h_z| - |h_{\overline{z}}|$ never vanish. Removing a relatively compact $W \subset V$ from Σ, we obtain an open Riemann surface $\Sigma \setminus \overline{W}$, which carries a strictly subharmonic exhaustion function ψ_0. We extend $\psi_0|\Sigma \setminus V$ to Σ so as to vanish on W, say ψ_1. We take $0 < c \ll 1$ for $\psi := c\psi_1$ to satisfy, in the all of the trivializing coordinates, $\psi_{z\overline{z}}(1-\delta)^{-3} > -1$, and

$$\psi_{z\overline{z}} + \frac{(1-\delta)^2}{4} \min\{|h_{\overline{z}}|^2, (|h_z| - |h_{\overline{z}}|)^2\} > 0 \quad \text{on } V.$$

Using this ψ, we have obtained the desired defining function r. □

Lemma 3.10. Let $\mathcal{D}, \Sigma$, and h as in Proposition 3.9. Then,

(i) The zero set of $h_{\overline{z}}$ is finite.
(ii) The set $\{|h_z| - |h_{\overline{z}}| \neq 0\}$ is open dense in Σ.

Proof. (i) We have well-defined forms $|h_z|(1-|h|^2)^{-1}|dz|$, $|h_{\overline{z}}(1-|h|^2)^{-1}|dz|$ and $\mathrm{Hopf}(h) := h_z\overline{h_{\overline{z}}}(1-|h|^2)^{-2}dz^2$ on Σ. The harmonicity of h is equivalent to holomorphicity of $\mathrm{Hopf}(h)$, whose zero set consists of $4g-4$ points. (Note that the assumption implies that $\pi_1(\Sigma)$ is non-abelian, thus genus of $\Sigma > 1$.) Therefore the zero set of $h_{\overline{z}}$ is also finite.

(ii) Suppose $\{|h_z| - |h_{\overline{z}}| = 0\} = \{\operatorname{rank} dh < 2\}$ contains a non-empty open set in Σ. From a theorem of Sampson [22, Theorem 3], the image of the lift $\tilde{h}\colon \widetilde{\Sigma} \to \mathbb{D}$ becomes a point, or a geodesic arc. The former case is impossible since the point is fixed by ρ and it is of the case (iv) in Theorem 3.1. The latter case is also impossible since the end points of the geodesic arc are fixed by ρ and it is of the case (iii) in Theorem 3.1. Thus, the claim is proved. □

Question 1. What about the case (iii)? We know an example in which $\mathcal{D}_\rho$ is Stein but not Takeuchi 1-complete ([16, Theorem 1.2]).

4. A Bochner-Hartogs type extension theorem

4.1. **A Bochner-Hartogs type extension theorem.** We will state a Bochner-Hartogs type extension theorem for CR sections of finite regularity, which can be obtained by established procedures as in [17], [3] and [5]. Here we give a simple proof for the reader's convenience. For the standard techniques used in this section, we refer the reader to the "OpenContent Book" of Demailly [6].

Theorem 4.1. *Let X be a connected compact complex manifold of dimension $n \geq 2$, L a holomorphic line bundle over X, and M a $\mathcal{C}^\infty$ compact Levi-flat real hypersurface of X which splits X into two Takeuchi 1-complete domains $D \sqcup D'$. Then, there exists $\kappa \in \mathbb{N}$ such that any $\mathcal{C}^\kappa$ CR section of $L|M$ extends to a holomorphic section of L.*

Proof. We set

$$N_0 := \min \left\{ N \in \mathbb{N} \;\middle|\; \begin{array}{l} i\Theta_{h_0} - Ni\partial\overline{\partial}(-\log(-r)) < 0 \quad \text{on } D, \\ i\Theta_{h_0} - Ni\partial\overline{\partial}(-\log(-r')) < 0 \quad \text{on } D', \\ h_0\text{: hermitian metric of } L, \\ r \text{ (resp. } r'\text{): defining function of } M \\ \quad \text{which makes } D \text{ (resp. } D'\text{) Takeuchi 1-complete} \end{array} \right\}.$$

The assumption yields $N_0 < \infty$. Put $\kappa := \lceil n+1+N_0/2 \rceil (\geq 4)$. Take h_0, r, and r' to attain the minimum, and fix an arbitrary hermitian metric g_0 of X.

We denote by $\langle \cdot, \cdot \rangle_{g_0,h_0}$ (resp. $|\cdot|_{g_0,h_0}$) the fiber metric (resp. norm) of $L \otimes \bigwedge \mathbb{C}TX^*$ determined by g_0 and h_0, and write $d\mathrm{vol}_{g_0}$ for the volume form on X determined by g_0. Integration with respect to these metrics is denoted by

$$\langle\!\langle \omega, \eta \rangle\!\rangle_{g_0,h_0,D} := \int_D \langle \omega, \eta \rangle_{g_0,h_0} d\mathrm{vol}_{g_0}$$

and write $\|\omega\|^2_{g_0,h_0,D} := \langle\omega,\omega\rangle_{g_0,h_0,D}$. We also use the following notation for function spaces.

- $\mathcal{C}^{\kappa}_{(p,q)}(X, L)$: the space of L-valued $\mathcal{C}^{\kappa}$ (p,q)-forms over X.
- $\mathcal{C}^{\kappa}_{0,(p,q)}(D, L)$: the space of L-valued compactly supported $\mathcal{C}^{\kappa}$ (p,q)-forms over D.
- $L^2_{(p,q)}(D, L; g_0, h_0)$: the space of L-valued measurable (p,q)-forms over D whose $\|\cdot\|_{g_0,h_0,D}$ norm is finite.

We will omit the subscript (p,q) when $(p,q) = (0,0)$.

The proof is separated into three lemmas.

Lemma 4.2. Let s be a $\mathcal{C}^{\kappa}$ CR section of $L|M$. Then we can extend s to $\tilde{s} \in \mathcal{C}^2(X, L)$ so that

$$|\overline{\partial}\tilde{s}|_0 := |\overline{\partial}\tilde{s}|_{g_0,h_0} = O(r^{\kappa-2}) \quad \text{along } M \tag{1}$$

where r is any $\mathcal{C}^{\infty}$ defining function of M.

Proof of Lemma 4.2. Firstly, we extend s to a $\mathcal{C}^{\kappa}$ section of L, still denoted by s, using a $\mathcal{C}^{\infty}$ collaring $M \times (-\epsilon, \epsilon) \to X$ of M and a transversal cut-off function with enough small support. Since $s|M$ is CR, we can find a $\mathcal{C}^{\kappa-1}$ section of $L|M$, say α_1, such that $\overline{\partial}s = \alpha_1 \overline{\partial}r$ on M. We extend α_1 to a $\mathcal{C}^{\kappa-1}$ section of L. Put $s_1 := s - \alpha_1 r$. Then, $|\overline{\partial}s_1|_0 = |(\overline{\partial}s - \alpha_1\overline{\partial}r) - \overline{\partial}\alpha_1 r|_0 = O(r)$ because $\overline{\partial}s_1$ vanishes on M and is of class $\mathcal{C}^{\kappa-2}$.

Suppose we have inductively constructed a $\mathcal{C}^{\kappa-\ell}$ extension s_ℓ of s with $s_\ell = s - \alpha_1 r - \alpha_2 r^2/2 - \cdots - \alpha_\ell r^\ell/\ell$ and $|\overline{\partial}s_\ell|_0 = O(r^\ell)$. Write $\overline{\partial}s_\ell = \beta_\ell r^\ell$ with $\beta_\ell \in \mathcal{C}^{\kappa-(\ell+1)}_{(0,1)}(X, L)$. We obtain $0 = \overline{\partial}^2 s_\ell = \overline{\partial}\beta_\ell r + \overline{\partial}r \wedge \beta_\ell$. Thus, we can find $\alpha_{\ell+1} \in \mathcal{C}^{\kappa-(\ell+1)}(X, L)$ such that $\beta_\ell = \alpha_{\ell+1}\overline{\partial}r$ on M. Putting $s_{\ell+1} := s_\ell - \alpha_{\ell+1}r^{\ell+1}/(\ell+1)$ gives $|\overline{\partial}s_{\ell+1}|_0 = |(\beta_\ell - \alpha_{\ell+1}\overline{\partial}r)r^\ell - \overline{\partial}\alpha_{\ell+1}r^{\ell+1}|_0 = O(r^{\ell+1})$ while $\beta_\ell - \alpha_{\ell+1}r^\ell$ is differentiable, which holds if $\kappa - (\ell+1) \geq 1$.

Letting $\tilde{s} := s_{\kappa-2}$ completes the proof. □

We perform a correction to $\tilde{s}$ to obtain the desired holomorphic extension. Once we solve the $\overline{\partial}$-equation $\overline{\partial}u = \overline{\partial}\tilde{s}$ on X in the distribution sense with the condition $u|M = 0$, we obtain the desired extension $\tilde{s} - u$ since holomorphic functions are characterized as weak solutions of the Cauchy-Riemann equation.

By Theorem 3.7, $i\partial\overline{\partial}(-\log(-r))$ defines a complete Kähler metric g on D, which blows up in $O(r^{-2})$ along M. Consider the hermitian metric $h = h_0 r^{-N_0}$ on L. The condition (1) on $\tilde{s}$ implies

$$\begin{aligned} \|\overline{\partial}\tilde{s}\|^2_{g,h} &:= \|\overline{\partial}\tilde{s}\|^2_{g,h,D} \\ &= \int_D |\overline{\partial}\tilde{s}|^2_{g,h} d\mathrm{vol}_g = \int_D O(r^{2(\kappa-2)})O(r^2)O(r^{-N_0})O(r^{-2n}) < \infty, \end{aligned}$$

i.e., $\overline{\partial}\tilde{s} \in L^2_{(0,1)}(D, L; g, h)$. We can solve $\overline{\partial}u = \overline{\partial}\tilde{s}$ on D thanks to the following L^2 cohomology vanishing theorem.

Lemma 4.3. For any $v \in L^2_{(0,1)}(D, L; g, h)$ with $\overline{\partial}v = 0$, there exists a solution $u \in L^2(D, L; g, h)$ of $\overline{\partial}u = v$ in the sense that there exists a sequence $u_n \in \mathcal{C}^\infty_0(D, L)$ such that $u_n \to u$ in $L^2(D, L; g, h)$ and $\overline{\partial}u_n \to v$ in $L^2_{(0,1)}(D, L; g, h)$.

Proof of Lemma 4.3. By the standard L^2 method of Andreotti-Vesentini [1], the conclusion follows from the following estimate

$$\|\overline{\partial}u\|^2_{g,h} + \|\overline{\partial}^*_{g,h}u\|^2_{g,h} \gtrsim \|u\|^2_{g,h}$$

for $u \in C^\infty_{0,(0,1)}(D, L)$. Here $\overline{\partial}^*_{g,h}$ denotes the formal adjoint of the operator $\overline{\partial}\colon L^2(D, L; g, h) \to L^2_{(0,1)}(D, L; g, h)$. Note that we have used the completeness of g to obtain the solution not only in the sense of distribution but also in the sense above.

By the Nakano inequality, we achieve the estimate as follows:

$$\begin{aligned}\|\overline{\partial}u\|^2_{g,h} + \|\overline{\partial}^*_{g,h}u\|^2_{g,h} &\gtrsim \langle\!\langle [i\Theta_h, \Lambda]u, u\rangle\!\rangle_{g,h} = -\langle\!\langle i\Theta_h u, Lu\rangle\!\rangle_{g,h}\\ &\gtrsim -\min\left\{\begin{array}{l}\text{sum of the } (n-1) \text{ eigenvalues of } i\Theta_h\\ \text{with respect to } g\end{array}\right\}\|u\|^2_{g,h}.\end{aligned}$$

The eigenvalues of $i\Theta_h$ with respect to g tend to $-N_0$ near M. It follows that the RHS $\gtrsim \|u\|^2_{g,h}$. □

Performing the same procedure on D', we obtain a section u of $L|D \sqcup D'$ with $\overline{\partial}u = \overline{\partial}\tilde{s}$ on $D \sqcup D'$ in the sense above. Consider the zero extension of u on X, still denoted by u. The following lemma completes the proof of Theorem 4.1.

Lemma 4.4. $\overline{\partial}u = \overline{\partial}\tilde{s}$ on X in the sense of distribution.

Proof. Let $u_n \in \mathcal{C}^\infty_0(D \sqcup D', L) \subset \mathcal{C}^\infty(X, L)$ be the approximation of u found in Lemma 4.3. Since $L^2(D, L; g_0, h_0) \hookrightarrow L^2(D, L; g, h)$ is continuous, we have $u_n \to u$ in $L^2(D \sqcup D', L; g_0, h_0) \simeq L^2(X, L; g_0, h_0)$.

Take a test function $\phi \in \mathcal{C}^\infty(X, L)$. Denote by $\overline{\partial}^*_0$ the formal adjoint of the operator $\overline{\partial}\colon L^2(X, L; g_0, h_0) \to L^2_{(0,1)}(X, L; g_0, h_0)$. Then,

$$\begin{aligned}\langle\!\langle \overline{\partial}u - \overline{\partial}\tilde{s}, \phi\rangle\!\rangle_{g_0,h_0,X} &= \langle\!\langle u, \overline{\partial}^*_0\phi\rangle\!\rangle_{g_0,h_0,X} - \langle\!\langle \overline{\partial}\tilde{s}, \phi\rangle\!\rangle_{g_0,h_0,X}\\ &= \lim_{n\to\infty}\langle\!\langle u_n, \overline{\partial}^*_0\phi\rangle\!\rangle_{g_0,h_0,X} - \langle\!\langle \overline{\partial}\tilde{s}, \phi\rangle\!\rangle_{g_0,h_0,X}\\ &= \lim_{n\to\infty}\langle\!\langle u_n, \overline{\partial}^*_0\phi\rangle\!\rangle_{g_0,h_0,D\sqcup D'} - \langle\!\langle \overline{\partial}\tilde{s}, \phi\rangle\!\rangle_{g_0,h_0,D\sqcup D'}\\ &= \lim_{n\to\infty}\langle\!\langle \overline{\partial}u_n - \overline{\partial}\tilde{s}, \phi\rangle\!\rangle_{g_0,h_0,D\sqcup D'}\\ &= 0.\end{aligned}$$

It completes the proof. □

□

Corollary 4.5. Suppose X, L, M, and κ as in Theorem 4.1. Then, all of the $\mathcal{C}^\kappa$ CR sections of $L|M$ are automatically of class $\mathcal{C}^\infty$, and they form a finite dimensional vector space.

We will use the following form of Theorem 4.1 in the proof of Main Theorem.

Corollary 4.6. Suppose X, L and M as in Theorem 4.1. Then, any $\mathcal{C}^\infty$ CR section of $L|M$ extends to a holomorphic section of L.

5. Conclusion

5.1. Proof of Main Theorem.

Proof of Main Theorem. From Proposition 3.9, $\mathcal{D}$ is Takeuchi 1-complete. The harmonic section of $X \setminus \overline{\mathcal{D}}$ is obtained by conjugating the harmonic section of $\mathcal{D}$. Thus, $X \setminus \overline{\mathcal{D}}$ is also Takeuchi 1-complete. Hence, Corollary 4.6 implies that for any $n \geq 1$, all of the $\mathcal{C}^\infty$ CR sections of $(\pi^* L|M)^{\otimes n}$ extend to holomorphic sections of $(\pi^* L)^{\otimes n}$.

On the other hand, $\pi^* \colon H^0(\Sigma, L^{\otimes n}) \to H^0(X, (\pi^* L)^{\otimes n})$ gives an isomorphism. Since we can give a trivializing cover of $(\pi^* L)^{\otimes n}$ by pulling back that of L, and the sections should be constant along any fiber $\pi^{-1}(p) \simeq \mathbb{CP}^1$ in these trivializations. Hence it is impossible for the sections in $H^0(X, (\pi^* L)^{\otimes n})$ to separate points in the same fiber for any n. Therefore, we cannot make a projective embedding by any ratio of those sections. □

5.2. Further questions. We conclude this paper with further questions.

Question 2. Can we prove Main Theorem intrinsically, i.e., without looking the natural Stein filling?

Question 3. Let M be a compact Levi-flat CR manifold, and L a CR line bundle over M. We define the threshold regularity $\kappa(M, L)$ to be the minimal $\kappa \in \mathbb{N} \cup \{\infty\}$, if exists, so that $\mathcal{C}^\kappa$ CR sections of L form a finite dimensional vector space. In the situation illustrated in Main Theorem, the proof of Theorem 4.1 indicates that $\kappa(M, (\pi^* L|M)^{\otimes n})$ is well-defined and $\kappa(M, (\pi^* L|M)^{\otimes n}) = O(n)$ as $n \to \infty$. On the other hand, Ohsawa-Sibony's projective embedding theorem implies that $\kappa(M, (\pi^* L|M)^{\otimes n}) \to \infty$ as $n \to \infty$. Can we read any dynamical property of the Levi foliation from the asymptotic behavior of the $\kappa(M, (\pi^* L|M)^{\otimes n})$?

Acknowledgments. The author would like to express his profound gratitude to his advisor T. Ohsawa, and is grateful to R. Kobayashi and K. Matsumoto for helpful comments which improved the presentation of the paper. Part of this work was done during “Ecole d’été 2012: Feuilletages, courbes pseudoholomorphes, applications” at Institute Fourier, and “Nagoya-Tongji joint workshop on Bergman kernel” at Tongji University. The author is grateful to both institutes for their support.

References

[1] Aldo Andreotti and Edoardo Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 81–130. MR0175148 (30 #5333)

[2] David E. Barrett, *Global convexity properties of some families of three-dimensional compact Levi-flat hypersurfaces*, Trans. Amer. Math. Soc. **332** (1992), no. 1, 459–474, DOI 10.2307/2154042. MR1055805 (93c:32026)

[3] Judith Brinkschulte, *The $\overline{\partial}$-problem with support conditions on some weakly pseudoconvex domains*, Ark. Mat. **42** (2004), no. 2, 259–282, DOI 10.1007/BF02385479. MR2101387 (2005f:32064)

[4] Marco Brunella, *On the dynamics of codimension one holomorphic foliations with ample normal bundle*, Indiana Univ. Math. J. **57** (2008), no. 7, 3101–3113, DOI 10.1512/iumj.2008.57.3394. MR2492227 (2010g:32051)

[5] Debraj Chakrabarti and Mei-Chi Shaw, *L^2 Serre duality on domains in complex manifolds and applications*, Trans. Amer. Math. Soc. **364** (2012), no. 7, 3529–3554, DOI 10.1090/S0002-9947-2012-05511-5. MR2901223

[6] Jean-Pierre Demailly, *Complex Analytic and Differential Geometry*, 2009.

[7] Bertrand Deroin, *Laminations dans les espaces projectifs complexes*, J. Inst. Math. Jussieu **7** (2008), no. 1, 67–91, DOI 10.1017/S1474748007000175 (French, with English and French summaries). MR2398147 (2009e:32029)

[8] Klas Diederich and Takeo Ohsawa, *Harmonic mappings and disc bundles over compact Kähler manifolds*, Publ. Res. Inst. Math. Sci. **21** (1985), no. 4, 819–833, DOI 10.2977/prims/1195178932. MR817167 (87g:32017)

[9] ———, *On the displacement rigidity of Levi flat hypersurfaces—the case of boundaries of disc bundles over compact Riemann surfaces*, Publ. Res. Inst. Math. Sci. **43** (2007), no. 1, 171–180. MR2319541 (2008m:32023)

[10] Yakov M. Eliashberg and William P. Thurston, *Confoliations*, University Lecture Series, vol. 13, American Mathematical Society, Providence, RI, 1998. MR1483314 (98m:53042)

[11] John Erik Fornæss and Erlend Fornæss Wold, *Solving $\bar\partial_b$ on hyperbolic laminations* (2011), available at `arXiv:1108.2286[math.CV]`.

[12] Étienne Ghys, *Laminations par surfaces de Riemann*, Dynamique et géométrie complexes (Lyon, 1997), Panor. Synthèses, vol. 8, Soc. Math. France, Paris, 1999, pp. ix, xi, 49–95 (French, with English and French summaries). MR1760843 (2001g:37068)

[13] Misha Gromov, *Topological invariants of dynamical systems and spaces of holomorphic maps. I*, Math. Phys. Anal. Geom. **2** (1999), no. 4, 323–415, DOI 10.1023/A:1009841100168. MR1742309 (2001j:37037)

[14] Phillip S. Harrington and Mei-Chi Shaw, *The strong Oka's lemma, bounded plurisubharmonic functions and the $\overline{\partial}$-Neumann problem*, Asian J. Math. **11** (2007), no. 1, 127–139. MR2304586 (2009m:32061)

[15] Takashi Inaba, *On the nonexistence of CR functions on Levi-flat CR manifolds*, Collect. Math. **43** (1992), no. 1, 83–87. MR1214226 (94i:32009)

[16] Takeo Ohsawa and Nessim Sibony, *Bounded p.s.h. functions and pseudoconvexity in Kähler manifold*, Nagoya Math. J. **149** (1998), 1–8. MR1619572 (2000b:32062)

[17] Takeo Ohsawa, *Pseudoconvex domains in $\mathbf{P}^n$: a question on the* 1*-convex boundary points*, Analysis and geometry in several complex variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 239–252. MR1699848 (2000j:32023)

[18] Takeo Ohsawa and Nessim Sibony, *Kähler identity on Levi flat manifolds and application to the embedding*, Nagoya Math. J. **158** (2000), 87–93. MR1766573 (2001d:32055)

[19] Takeo Ohsawa, *On Projectively Embeddable Complex-Foliated Structures*, Publ. Res. Inst. Math. Sci. **48** (2012), no. 3, 735–747, DOI 10.2977/PRIMS/87. MR2973399

[20] Kiyosi Oka, *Sur les fonctions analytiques de plusieurs variables. VI. Domaines pseudoconvexes*, Tôhoku Math. J. **49** (1942), 15–52 (French). MR0014470 (7,290a)

[21] Hansklaus Rummler, *Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts*, Comment. Math. Helv. **54** (1979), no. 2, 224–239, DOI 10.1007/BF02566270 (French). MR535057 (80m:57021)

[22] J. H. Sampson, *Some properties and applications of harmonic mappings*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 2, 211–228. MR510549 (80b:58031)

[23] Dennis Sullivan, *A homological characterization of foliations consisting of minimal surfaces*, Comment. Math. Helv. **54** (1979), no. 2, 218–223, DOI 10.1007/BF02566269. MR535056 (80m:57022)

[24] Akira Takeuchi, *Domaines pseudoconvexes infinis et la métrique riemannienne dans un espace projectif*, J. Math. Soc. Japan **16** (1964), 159–181 (French). MR0173789 (30 #3997)

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